NOTE ON GREGORY'S DISCUSSION OF THE CONDITIONS FOR AN UMBILICUS.

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Gregory's treatment of the conditions for an umbilicus in Arts. 280, 292, 295, 296* would gain in simplicity by a rearrangement of the material. It will be seen that the author obtains in Art. 280 the quadratic equation for the greatest and least values of the radius of curvature at a given point on the surface, and thence derives in Art. 292 the general conditions for an umbilicus, and hence the special conditions to be satisfied at points where any one of the three partial derivatives U, V, W vanishes. But in Art. 295† it is stated that "when any of the quantities U, V, or W is equal to zero, the transformations of Art. (280) . . . is [sic] impossible, or rather nugatory"!! Again, Art. 296 says: "In the preceding Article it has been remarked that, although in certain cases the investigation of Art. 280 involves indeterminate operations, yet the equation (18) there obtained will be universally true, and that it must be essentially the same as if it had been found by a process involving no nugatory expressions. We shall, however, now give an investigation which is at all times free from nugatory operations, in order that every light may be thrown upon this important equation." Then follows a proof of equation (18) which if substituted for that given in Art. 280 would render unnecessary the latter half of each of the Arts. 292, 295. For since the quadratic in question (now derived by a method free from any indeterminateness) becomes when $U = 0$, $V \gtrless 0$, $W \gtrless 0$

$$\left[\frac{P}{\rho} - u\right]\left[\frac{P}{\rho} - \frac{V^2 w + W^2 v - 2\,VWu'}{V^2 + W^2}\right] = \frac{(Vv' - Ww')^2}{V^2 + W^2}\,,‡$$

and since the necessary and sufficient conditions that an equation of the form $(z - a)(z - b) = c^2$ may have equal roots are $c = 0$, and $a = b$, hence the conditions for an umbilicus in this case are

$$Vv' - Ww' = 0\,, \quad u(V^2 + W^2) = V^2 w + W^2 v - 2\,VWu'\,.$$

Similarly may be treated other cases, such as $V = 0$, $U \gtrless 0$, $W \gtrless 0$; or $U = V = 0$, $W \gtrless 0$, etc.

When U, V, W are each not zero we may proceed according to the methods of Arts. 280, 292 (first half), and obtain formula (26), p. 264.

* A Treatise on the Application of Analysis to Solid Geometry by Gregory and Walton (Second Edition, 1852).

† Misprinted "225."

‡ See Gregory, pp. 268-9.